

# Fast Algorithms for Constructing Maximum Entropy Summary Trees

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**Abstract.** Karloff and Shirley recently proposed “summary trees” as a new way to visualize large rooted trees (Eurovis 2013) and gave algorithms for generating a maximum-entropy  $k$ -node summary tree of an input  $n$ -node rooted tree. However, the algorithm generating optimal summary trees was only pseudo-polynomial (and worked only for integral weights); the authors left open existence of a polynomial-time algorithm. In addition, the authors provided an additive approximation algorithm and a greedy heuristic, both working on real weights.

This paper shows how to construct maximum entropy  $k$ -node summary trees in time  $O(k^2n + n \log n)$  for *real* weights (indeed, as small as the time bound for the greedy heuristic given previously); how to speed up the approximation algorithm so that it runs in time  $O(n + (k^4/\epsilon) \log(k/\epsilon))$ , and how to speed up the greedy algorithm so as to run in time  $O(kn + n \log n)$ . Altogether, these results make summary trees a much more practical tool than before.

## 1 Introduction

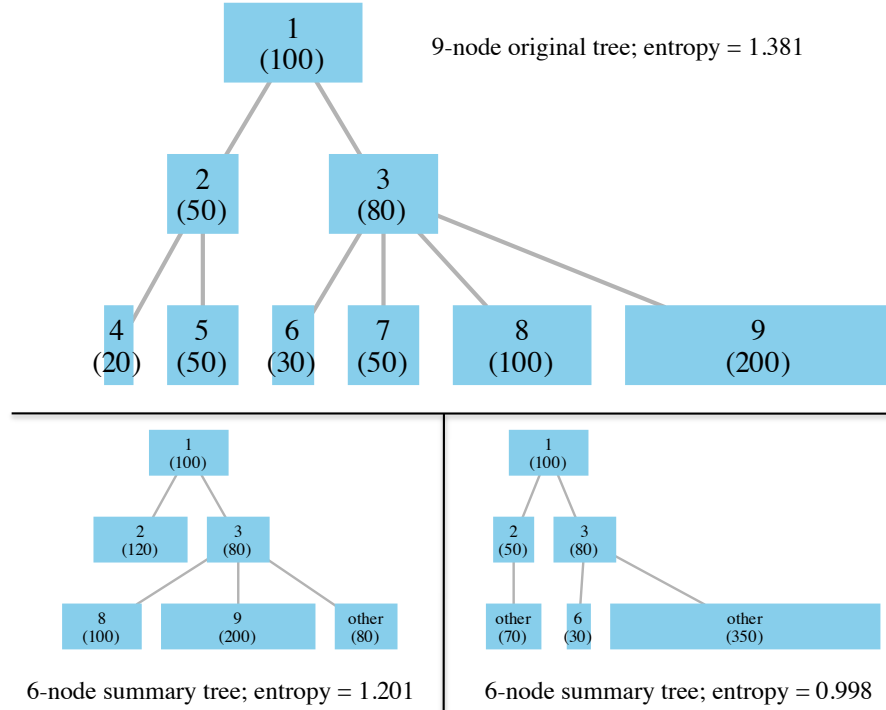
How should one draw a large  $n$ -node rooted tree on a small sheet of paper or computer screen? Recently, in Eurovis 2013, Karloff and Shirley [4] proposed a new way to visualize large trees. While the best introduction to summary trees appears in [4], here we give a necessarily short description. A user has an  $n$ -node node-weighted tree  $T$  and wants to draw a  $k$ -node summary  $S$  of  $T$  on a small screen or sheet of paper,  $k$  being user-specified. We begin with an informal, bottom-up, operational description. Two types of contraction are performed: subtrees are contracted to single nodes that represent the corresponding subtrees; similarly multiple sibling subtrees (subtrees whose roots are siblings) are contracted to single nodes representing them. The node resulting from the latter contraction is called a group node. The one constraint is that each node in the summary tree have at most one child that is a group node. Examples are shown in Figure 1–3 below (these figures appeared originally in [4]).

Next, we give a more formal description. Let  $T_v$  denote the subtree of  $T$  rooted at  $v$ . We name each node of  $S$  by the set of nodes of  $T$  that it represents.

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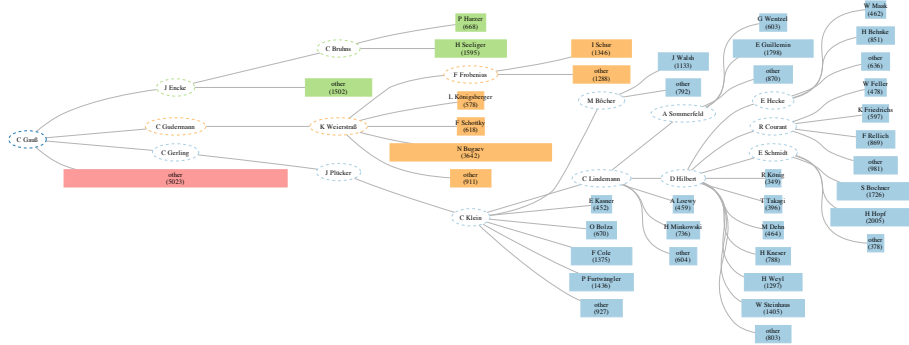
**Fig. 1.** In the upper panel, a 9-node tree (with node weights in parentheses), and below it, two different 6-node summary trees of the original 9-node tree.

The following comprise the possible *summary trees* for  $T_v$ : If  $T_v$  has just one node, the only summary tree is the one node  $\{v\}$ . Otherwise, a summary tree for  $T_v$  is one of:

1. a one-node tree  $V(T_v)$  (the set of nodes in  $T_v$ ); or
2. a singleton node  $\{v\}$  and summary trees for the subtrees rooted at the children of  $v$  (and edges from  $\{v\}$  to the roots of these summary trees); or
3. a singleton node  $\{v\}$ , a node *other<sub>v</sub>* representing a non-empty subset  $U_v$  of  $v$ 's children and all the descendants of the nodes  $x \in U_v$ , and for each of  $v$ 's children  $x \notin U_v$  a summary tree for  $T_x$  (and edges from  $\{v\}$  to *other<sub>v</sub>* and to the roots of the summary trees for each  $T_x$ ).<sup>1</sup> Sometimes we will overload the term *other<sub>v</sub>* by using it to denote the subset  $U_v$ .

We allow arbitrary nonnegative real weights  $w_v$  on the nodes  $v$  of the input tree  $T$ . The weight of a node in a summary tree is defined to be the sum of the weights of the corresponding nodes in  $T$ . Paper [4] defined the entropy of a

<sup>1</sup> *other<sub>v</sub>* sets of size 1 are covered by Cases 2 and 3, but this redundancy is convenient for the algorithm description.



**Fig. 2.** Taken from [4], the maximum entropy 56-node summary tree of the math genealogy tree rooted at Carl Friedrich Gauss, which has 43,527 equal-weighted nodes (where the original advisor-student graph was forced to be a tree by choosing the primary advisor for each student who had multiple advisors). Node colors are determined by their depth-1 ancestor, and node areas are proportional to their weights in the summary tree. This tree is best viewed (and enlarged) on a computer screen.

$k$ -node summary tree with nodes of weights  $W_1, W_2, \dots, W_k$  to be  $-\sum_{i=1}^k p_i \lg p_i$ , where  $p_i = W_i/W$  and  $W$  is the sum of all node weights, the usual information-theoretic entropy. Paper [4] then proposed that the most informative summary trees are those of maximum entropy. As noted in [4], this is a natural way to think about the information contained in a node-weighted tree. For given a bound on the number of nodes available in a summary tree, it seems plausible that a best summary tree is one of maximum entropy, because it is theoretically the most informative. This provided a principled way to identify the best  $k$ -node summary tree, in contrast to more heuristic and operational rules in prior work.

The fact that  $other_v$  is an *arbitrary* non-empty subset of  $v$ 's potentially large set of children is what makes finding maximum entropy summary trees difficult. Indeed, [4] resorted to using a dynamic program over the node weights (which worked provided that the weights were integral) and which led to a final running time of  $O(K^2 n W)$ , where  $W$  is the sum of the node weights and  $K$  is the maximum  $k$  for which one is interested in finding a  $k$ -node summary tree. Given  $K$ , the dynamic program finds maximum entropy  $k$ -node summary trees for  $k = 1, 2, \dots, K$ ; from now on we assume that the user specifies  $K$  and  $k$ -node summary trees are found for all  $k \leq K$ . The algorithm worked well when  $W$  was small, but failed to terminate on two of the five data sets used in [4].

The key to obtaining a running time independent of  $W$  is to develop a fuller understanding of the structure of maximum entropy summary trees. Our new understanding readily yields a truly polynomial-time algorithm. The main remaining challenge is to create and analyze an effective implementation. We give

	Optimal Entropy	Greedy	$\epsilon$ -Approximate
Known results	$O(K^2nW)$ [4]	$O(K^2n + n \log n)$ [4]	$O(K^2nW_0)$ [4]
New results	$O(K^2n + n \log n)$	$O(Kn + n \log n)$	$O(n + K^3W_0 + W_0 \log W_0)$

**Table 1.** Running times of the algorithms;  $W_0 = O((K/\epsilon) \log(K/\epsilon))$ .

an algorithm running in time  $O(K^2n + n \log n)$ <sup>2</sup>; it generates maximum entropy summary trees even for real weights, assuming, of course, a real-arithmetic model of computation, which is necessary (even for integral weights) because of the computation of logarithms. This result is based on a structural theorem which shows that the *other* sets, while allowed to be arbitrary, can be assumed, without loss of generality, to have a simple structure.

To deal with the case of real weights or exceedingly large integral weights, [4] gave an algorithm based on scaling, rounding, and algorithmic discrepancy theory which builds a summary tree whose entropy is within  $\epsilon$  *additively* of the maximum, in time  $O(K^2nW_0)$ , where  $W_0$  is  $O((K/\epsilon) \log(K/\epsilon))$ . Keep in mind here that  $K$  is meant to be small, e.g., 100 or 500, while  $n$  is meant to go to infinity, and also that  $W_0$  is a function only of  $K$  and  $\epsilon$  (and of neither  $n$  nor  $W$ ). The key here was to show that scaling the real input weights to have sum  $W_0$ , rounding them using algorithmic discrepancy theory, and then running the exact dynamic program previously mentioned on the rounded weights caused a loss of only  $\epsilon$  in the final entropy.

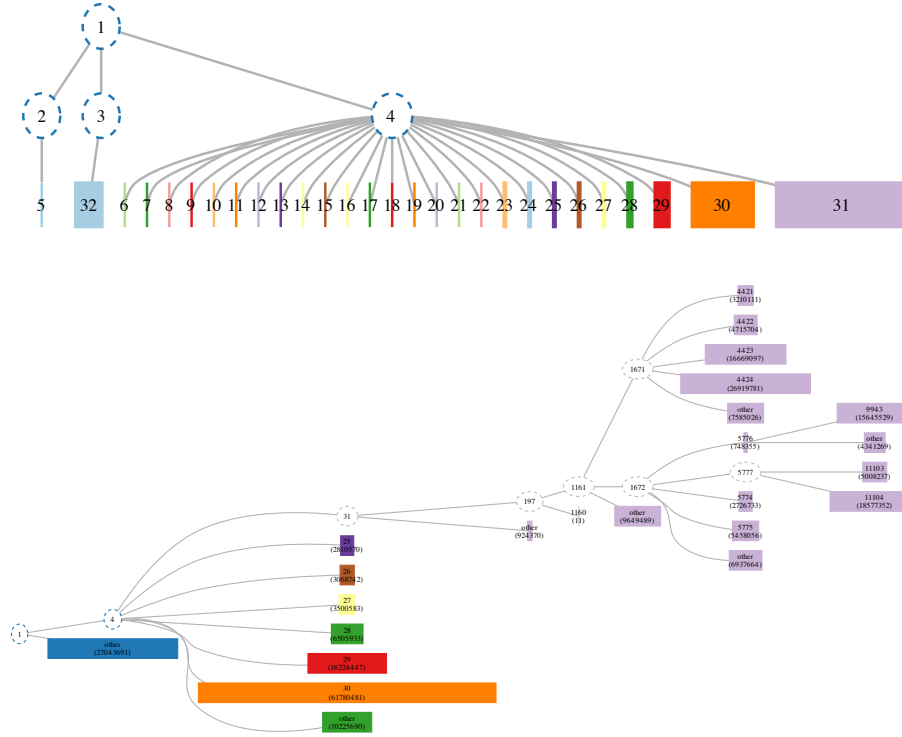
This paper shows that the same algorithm can be implemented in time  $O(n + K^3W_0 + W_0 \log W_0)$ ; this is linear time if  $n$  is larger than the other terms. The key here is to notice that if the sum of integral weights is  $W_0$ , which is small, and  $n \gg W_0$ , then most nodes have rounded weight 0. Surely one shouldn't have to devote a lot of time to nodes of weight 0, and our algorithm, by effectively replacing  $n$  by  $O(W_0)$ , exploits this intuition.

Last, [4] proposed a fast greedy algorithm to generate summary trees. Running in time  $O(K^2n + n \log n)$  (though [4] overlooked the  $n \log n$  time needed for sorting), the algorithm never took longer than six seconds to run on the data sets of [4]. This paper shows that a simple modification to the greedy code, neither suggested in [4] nor implemented in the associated C code, specifically, not computing a  $k$ -node summary tree of a tree rooted at a node having fewer than  $k$  descendants, decreases the running time bound of the greedy algorithm from  $O(K^2n + n \log n)$  to  $O(Kn + n \log n)$ . While the modification is trivial, its analysis is not.

Taken together, these new results show that maximum entropy summary trees are a much more practical tool than was previously known.

*Roadmap.* Section 2 describes earlier work on visualizing trees. In Section 3 we prove the structural theorem on which our improved algorithms depend. This

<sup>2</sup> Actually, this can be reduced to  $O(K^2n)$  time by using a combination of fast selection and sorting instead of sorting alone in various places.



**Fig. 3.** Two summary trees of a 19,335-node web traffic tree. The upper figure is a naive aggregation to depth 2; the node weights are heavily skewed. The bottom figure is the maximum entropy 32-node summary tree, which displays much more information given the same number of nodes.

is followed in Section 4 with our exact algorithm and in Section 5 with the key lemma for analyzing the exact and greedy algorithms. Section 6 gives the greedy algorithm and Section 7, the approximate algorithm and its analysis.

## 2 Previous Work

Traditionally tree visualization involved either visualizing the entire tree or allowing the user to interactively specify in what part of the tree he or she is interested. Obviously, if one draws a huge tree on a sheet of paper or a computer screen, not only will labels be close-to-impossible to read, there will be too much information, in that the reader will not know on what part to focus.

Many researchers have attempted to ameliorate the issues involved with drawing a huge tree by allowing interactivity. Initially perhaps only the root of the tree is displayed. When the user clicks on a node, that node’s children then appear. “Degree-of-interest trees” [2, 3] let a user explore a tree interactively.

Other interactive techniques are “hyperbolic browsers” [5] and the “accordion drawing technique” [7, 1].

Researchers have proposed “space-filling” layouts as an alternative to traditional node-and-edge layouts. Treemaps [9] are one popular way to lay out large trees. The root node is represented by a rectangle, and recursively the children of a node  $v$  are represented by rectangles which together partition the rectangle representing  $v$ . But treemaps are not effective at showing the hierarchy of a tree.

Von Landesberger et al. wrote a recent survey [6] on techniques for drawing large graphs. Other relevant previous work can be found in [4].

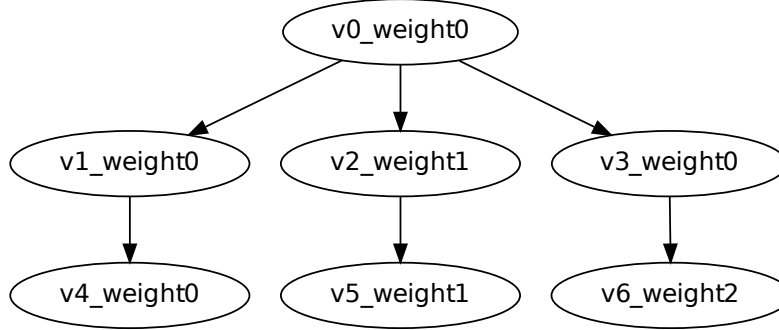
### 3 Structural Theorem

This section proves a structural theorem which implies that maximum entropy summary trees can be computed in polynomial time, in a real-arithmetic model of computation. We begin by relating our approach to the greedy algorithm from [4]. Let  $v$  be a node of an input tree and suppose that  $\{v\}$  appears in the summary tree. Recall that  $other_v$  denotes the group child of  $v$ , if any.

**Definition 1.** 1. The size  $s_v$  of a node  $v$  in  $T$  is the sum of the weights of its descendants.  
 2.  $n_v$  denotes the number of descendants of  $v$  (including  $v$ ).  
 3.  $d_v$  denotes the degree of  $v$ , the number of children it has.  
 4.  $\langle v_1, v_2, \dots, v_{d_v} \rangle$  denotes the children of  $v$  when sorted into nondecreasing order by size. (Fix one sorted order for each  $v$ , breaking ties arbitrarily.)  
 5. The prefixes of  $\langle v_1, v_2, \dots, v_{d_v} \rangle$  are the sequences  $\langle v_1, v_2, \dots, v_i \rangle$  and sets  $\{v_1, v_2, \dots, v_i\}$  for  $i \geq 0$ .

The greedy algorithm in [4] sorted and then processed the children of each node in nondecreasing order by size; more about this later. It finds a maximum entropy summary tree among those in which for each  $v$ , either  $other_v$  does not exist or is a nonempty prefix of  $\langle v_1, v_2, \dots, v_{d_v} \rangle$ , but this need not be the optimal summary tree. In fact, [4] gives a 7-node tree  $T$  for which the uniquely optimal 4-node summary tree has an  $other_v$  node which is not a prefix of  $v$ ’s children (see Figure 4). In their example, the greedy algorithm achieves approximately 1 bit of entropy, but the optimal summary tree achieves approximately 1.5 bits (and 1.5/1.0 is the worst ratio between greedy and optimum of which we are aware). This example proves that restricting  $other_v$  to be a prefix of the list of  $v$ ’s children can lead to summary trees of suboptimal entropy. Consequently, [4] resorted to a pseudo-polynomial-time dynamic program in order to find the optimal  $other$  sets.

The definition of summary trees allows  $other_v$  to represent an *arbitrary* nonempty subset of  $v$ ’s children (and all their descendants). However, in this paper we prove the surprising fact that, without loss of generality, in every summary tree of *maximum entropy*,  $other_v$  can be assumed to have a special form, a simple extension of the “prefix” form used in the greedy algorithm from [4].



**Fig. 4.** A 7-node tree on which the greedy algorithm does particularly badly. Imagine that the weight of node  $v_6$  slightly exceeds 2, so that the unique sorted order of the root's children into nondecreasing order by size is  $\langle v_1, v_2, v_3 \rangle$ . The unique 4-node maximum entropy summary tree has  $other_{v_0} = \{v_1, v_3\}$ , which is not a prefix of  $\langle v_1, v_2, v_3 \rangle$ ; this summary tree has entropy 1.5. By contrast, greedy gets  $other_{v_0} = \{v_1, v_2\}$ , in a summary tree of entropy 1.

**Definition 2.** The near-prefixes of  $\langle v_1, v_2, \dots, v_{d_v} \rangle$  are the sequences  $\langle v_1, v_2, \dots, v_i; v_j \rangle$  and the sets  $\{v_1, v_2, \dots, v_i; v_j\}$  where  $i \geq 0$ ,  $j \geq i + 2$ , and  $j \leq d_v$ .  $v_j$  is called the non-prefix element. This terminology is also applied to the sequence  $\langle T_{v_1}, T_{v_2}, \dots, T_{v_{d_v}} \rangle$  of trees rooted at  $v_1, v_2, \dots, v_{d_v}$ , respectively.

We prove the following structural theorem:

**Theorem 1.** For each  $k$ ,  $1 \leq k \leq n$ , there is a maximum entropy  $k$ -node summary tree  $S$  in which, for every node  $v$ ,  $other_v$ , when present, is either a prefix or a near-prefix of  $\langle T_{v_1}, T_{v_2}, \dots, T_{v_{d_v}} \rangle$ .

*Proof.* For any summary tree  $R$  of an  $n$ -node tree  $T$ , let  $M = 2n + 1$  and define  $\Phi(R) = \sum_{v: other_v \text{ exists}} M^{n-d_R(v)} \sum_{j: v_j \in other_v} j$ , where  $d_R(v)$  denotes the depth in  $R$  of the node  $other_v$ . Among all maximum entropy summary trees for  $T$ , let  $S$  be one for which  $\Phi(S)$  is minimum. (The role of  $\Phi$  will be to enable tie-breaking among equal-weight summary trees.)

**Lemma 1.** Let  $v$  be a node of  $T$  such that  $other_v$  exists in  $S$ . If  $v_i \notin other_v$  and  $v_j \in other_v$ , where  $i < j$ , then  $T_{v_i}$  is represented by two or more nodes in  $S$ .

*Proof.* Suppose, for a contradiction, that  $T_{v_i}$  is represented by a single node. Consider the following alternate summary tree  $S'$ :  $S'$  is obtained from  $S$  by replacing  $v_j$  in  $other_v$  by  $v_i$ , and by representing  $T_{v_j}$  by a single node. The number of nodes in the summary tree remains  $k$ .

Let  $s_0$  denote the sum of the sizes of all the children of  $v$  in  $other_v - \{v_j\}$ . (Here “ $other_v$ ” refers to  $other_v$  before the change.) Then  $W$  times the increase in entropy in going from  $S$  to  $S'$  is given by

$$I = (s_0 + s_{v_i}) \lg \frac{W}{s_0 + s_{v_i}} + s_{v_j} \lg \frac{W}{s_{v_j}} - (s_0 + s_{v_j}) \lg \frac{W}{s_0 + s_{v_j}} - s_{v_i} \lg \frac{W}{s_{v_i}}.$$

The derivative of this term with respect to  $s_{v_i}$  is  $\lg \frac{s_{v_i}}{s_0 + s_{v_i}} \leq 0$ . As  $i < j$ ,  $s_{v_i} \leq s_{v_j}$ , and thus  $I$  is necessarily nonnegative (for it declines to 0 at  $s_{v_i} = s_{v_j}$ ); consequently, there is a nonnegative increase in entropy, and hence  $S'$  is also a maximum entropy summary tree. Furthermore, if  $d$  is the depth of  $other_v$  in  $S$ , then  $\Phi(S') - \Phi(S) \leq -(j - i)M^{n-d} < 0$ , which contradicts the assumption that  $S$  is a maximum entropy summary tree of minimum  $\Phi(S)$ . ■

**Lemma 2.** *Let  $v$  be a node in  $T$  such that  $other_v$  exists in  $S$ . If  $v_i \notin other_v$  and  $v_{i+1} \in other_v$ , then  $v_j \notin other_v$  for all  $j > i + 1$ .*

*Proof.* Suppose, for a contradiction, that  $v_j \in other_v$ , for some  $j > i + 1$ .

By Lemma 1,  $T_{v_i}$  is represented by two or more nodes in  $S$ . Hence  $\{v_i\}$  appears as a node in the summary tree, and  $\{v_i\}$  has one or more children in  $S$ . In  $S$ , let  $x$  be a descendant of  $\{v_i\}$  of maximum depth in  $S$ . Node  $x$  is a proper descendant of  $\{v_i\}$ .

We will show now that combining node  $x$  with another node in a specified way yields a summary tree of  $T_{v_i}$  with one fewer node and having entropy at most  $s_{v_i}$  smaller. Node  $x$  is not  $\{v_i\}$ . Let  $y$  be  $x$ 's parent in  $S$ . Node  $y = \{u\}$  for some node  $u$  in  $T$  (since every nonleaf in a summary tree represents a single node of  $T$ ). There are four cases to analyze, but before turning to them, we state the following simple lemma which we will need; it can be proven by calculus.

**Lemma 3.** *If  $a, b \geq 0$ ,  $-a \lg a - b \lg b + (a + b) \lg(a + b) \leq a + b$ .*

Let  $s_x$ , for a node  $x$  in summary tree  $S$ , denote the sum of the weights of all the nodes of  $T$  represented by  $x$ . (For a node of the form  $other_v$ , we mean the sum of the *sizes* of all the children of  $v$  in  $other_v$ , or equivalently, the sum of the weights of all their descendants.)

Now we begin the case analysis. Let  $d$  be the depth in  $S$  of node  $\{v_i\}$ .

1.  $y$ 's only child in  $S$  is  $x$ .

We combine nodes  $x$  and  $y = \{u\}$  into a node  $z$  representing  $T_u$ . Recall that  $w_u$  denotes  $u$ 's weight. Then  $W$  times the entropy decrease equals

$$\begin{aligned} & s_x \lg(W/s_x) + w_u \lg(W/w_u) - (s_x + w_u) \lg(W/(s_x + w_u)) \\ &= -s_x \lg s_x - w_u \lg w_u + (s_x + w_u) \lg(s_x + w_u) \\ &\leq s_x + w_u \quad (\text{by Lemma 3}) = s_z \leq s_{v_i}. \end{aligned}$$

This change leaves  $\Phi$  unchanged.



2.  $x$  has a sibling in  $S$  and  $other_u$  does not exist.

Hence  $x$  is either  $\{\alpha\}$  or  $T_\alpha$  for some node  $\alpha \in T$ .

We create a new  $other_u$  node by combining  $x$  with an arbitrary sibling  $x'$  of  $x$ . Because  $x$  is of maximum depth in  $S$ ,  $x'$  is either of the form  $\{\beta\}$  (node  $\beta$  in  $T$  has no children) or  $T_\beta$ , for some  $\beta$  in  $T$ . The resulting entropy decrease equals

$$\begin{aligned} & s_x \lg(W/s_x) + s_{x'} \lg(W/s_{x'}) - (s_x + s_{x'}) \lg(W/(s_x + s_{x'})) \\ &= -s_x \lg s_x - s_{x'} \lg s_{x'} + (s_x + s_{x'}) \lg(s_x + s_{x'}) \\ &\leq s_x + s_{x'} \quad (\text{by Lemma 3}) \leq s_{v_i}. \end{aligned}$$

This change can increase  $\Phi$  by at most  $2n \cdot M^{n-(d+1)}$ , because the depth of the new  $other_u$  node is at least  $d+1$ .

3.  $x$  has a sibling in  $S$  and  $\{x\} = other_u$ .

We choose an arbitrary sibling  $x'$  of  $x$  and add it to  $other_u$ . The entropy calculation is the same as for Case 2. This change can increase  $\Phi$  by at most  $n \cdot M^{n-(d+1)}$ , where  $d$  is the depth of  $\{v_i\}$  in  $S$ .

4.  $x$  has a sibling in  $S$ ,  $other_u$  exists, and  $\{x\} \neq other_u$ .

We add  $x$  to  $other_u$ . Let  $x'$  be the node  $other_u$ . The calculations are exactly the same as in Case 3.

In all four cases, the decrease in entropy is at most  $s_{v_i}$  and the increase in  $\Phi$  is at most  $2nM^{n-d-1}$ .

Now we show how to generate a new maximum entropy summary tree  $S'$ . To get  $S'$ , combine  $x$  as above with either its parent or a sibling, thereby decreasing the number of summary tree nodes by one, and then split off  $v_{i+1}$  from  $other_v$  and create a node to represent  $T_{v_{i+1}}$ , thereby increasing the number of summary tree nodes back to  $k$ . Now, let  $s_0$  denote the sum of the sizes of all the children of  $v$  in  $other_v - \{v_{i+1}, v_j\}$ .  $W$  times the increase in entropy from this two-part change to  $S$  is at least

$$\begin{aligned} & \left[ (s_0 + s_{v_j}) \lg \frac{1}{s_0 + s_{v_j}} + s_{v_{i+1}} \lg \frac{1}{s_{v_{i+1}}} - (s_0 + s_{v_{i+1}} + s_{v_j}) \lg \frac{1}{s_0 + s_{v_{i+1}} + s_{v_j}} \right] - s_{v_i} \\ &= (s_0 + s_{v_j}) \lg \frac{s_0 + s_{v_{i+1}} + s_{v_j}}{s_0 + s_{v_j}} + s_{v_{i+1}} \lg \frac{s_0 + s_{v_{i+1}} + s_{v_j}}{s_{v_{i+1}}} - s_{v_i} \geq s_{v_{i+1}} - s_{v_i} \geq 0. \end{aligned}$$

(The first inequality follows because  $s_{v_j} \geq s_{v_{i+1}}$ , which implies that  $(s_0 + s_{v_{i+1}} + s_{v_j})/s_{v_{i+1}} \geq 2$ .) But this is a nonnegative increase in entropy, proving that  $S'$  is a maximum entropy summary tree.

Splitting off  $v_{i+1}$  from  $other_v$  decreases  $\Phi$  by at least  $M^{n-d}$ , because the depth of the  $other_v$  node equals the depth of node  $v_i$ , which is  $d$ . Hence the total  $\Delta\Phi$  is at most  $-M^{n-d} + 2n \cdot M^{n-d-1} = -M^{n-d}(1 - 2n/M) < 0$ , a contradiction to the fact that  $S$  is a maximum entropy summary tree of minimum  $\Phi$ . ■

This completes the proof of Theorem 1. ■

**Theorem 2.** For all  $v$ , if  $other_v$  exists, then  $|other_v| \geq d_v - K + 2$ .

*Proof.* Each child of  $v$  not in  $other_v$  contributes at least one node to the final summary tree, which has order  $k \leq K$ , and hence the number of children not in  $other_v$  cannot exceed  $K - 2$  (for one node is needed to represent  $\{v\}$ ). ■

## 4 The Exact Algorithm

Relabel the nodes as  $1, 2, \dots, n$ , with the root being node 1, the nodes at depth  $d$  getting consecutive labels, and the children of a node being labeled with increasing consecutive labels in nondecreasing size order. (This can be done by processing the nodes in nondecreasing order by depth, with all the children of node  $v$  processed consecutively in nondecreasing order by size.) This relabeling costs  $O(n \log n)$  time,<sup>3</sup> because  $\sum_v (d_v \log d_v) \leq \sum_v (d_v \log n) \leq n \log n$ .

The description and the implementation of the algorithm are simplified if we compute what we call the “pseudo-entropy,” of summary trees for  $T_v$  rather than their entropy. The *pseudo-entropy*  $p\text{-ent}(S_v)$  of a tree  $S_v$  with nodes of weights  $W_1, W_2, \dots, W_k$  is simply  $-\sum p_i \log p_i$ , where  $p_i = W_i/W$  and  $W$  is the weight of  $T$  (and *not* of  $T_v$ ). Clearly, if  $S_v$  is part of a summary tree  $S$  for  $T$ , then  $S_v$  contributes  $-\sum p_i \log p_i$  to the entropy of  $S$ . Let  $\text{ent}(S_v)$  denote the entropy of tree  $S_v$ . Then

$$\begin{aligned} \text{ent}(S_v) &= -\sum_i \frac{W_i}{W_v} \log \frac{W_i}{W_v} = -\left[ \frac{W}{W_v} \sum_i \frac{W_i}{W} \log \frac{W_i}{W} + \sum_i W_i W \log \frac{W}{W_v} \right] \\ &= -\frac{W}{W_v} p\text{-ent}(S_v) - \log \frac{W}{W_v}. \end{aligned}$$

Thus the same tree optimizes the entropy and the pseudo-entropy.

We will be using a dynamic programming algorithm. To simplify the presentation we will only describe how to compute the maximum pseudo-entropy for a  $k$ -node summary tree for  $T_v$ , for each node  $v$  and for all  $k$ ,  $1 \leq k \leq \min\{K, n_v\}$ .

The algorithm will first seek to find the value of the pseudo-entropy for optimal  $k$ -node summary trees when  $other_v$  is restricted to being a prefix set, and then when  $other_v$  is restricted to being a near-prefix set containing  $v_j$  as its non-prefix element, for each possible  $v_j$  in turn, i.e., for  $\max\{3, d_v - K + 3\} \leq j \leq d_v$ . Thus the algorithm will consider  $\min\{d_v - 1, K - 1\} \min\{d_v, K - 1\}$  classes of candidate  $other_v$  sets.

To describe the algorithm it will be helpful to introduce the notion of a summary forest. A  $k$ -node *summary forest* for  $T_v$  is a  $(k+1)$ -node summary tree for  $T_v$  from which  $v$  has been excised (leaving a forest). We will also call this a summary forest for  $T_{v_1}, T_{v_2}, \dots, T_{v_{d_v}}$ . A summary forest for  $T_{v_1}, T_{v_2}, \dots, T_{v_l}$  is defined analogously, for  $1 \leq l \leq d_v$ .

<sup>3</sup> In fact, the relative order, at node  $v$ , of its  $d_v - K + 1$  smallest-sized children does not matter since they must all be included in  $other_v$ . This allows us to perform just a partial sort at each node, in which the  $d_v - K + 1$  smallest-size children are identified by selection and then the remaining at most  $K - 1$  children are sorted. This improves the  $O(n \log n)$  term to  $O(n \log K)$  which is dominated by  $O(nK)$ .

To find the pseudo-entropy-optimal  $k$ -node summary trees for  $T_v$ , for  $1 \leq k \leq K$ , we first find the pseudo-entropy of optimal  $k$ -node summary forests for  $T_{v_1}, T_{v_2}, \dots, T_{v_l}$ , for  $\max\{1, d_v - K + 2\} \leq l \leq d_v$ . The optimal  $k$ -node summary trees for  $T_v$  are then obtained by attaching  $\{v\}$  as a root node to the trees in the optimal  $(k - 1)$ -node summary forests for  $T_{v_1}, T_{v_2}, \dots, T_{v_{d_v}}$ .

Now we explain how to find these optimal summary forests. In turn, we consider each of the up-to- $\max\{1, K - 1\}$  possible classes of *other<sub>v</sub>* nodes: the prefix *other<sub>v</sub>* nodes, and for each  $j$  with  $\max\{3, d_v - K + 3\} \leq j \leq d_v$ , the class of near-prefix *other<sub>v</sub>* nodes including  $v_j$  as the non-prefix element.

First, we describe the handling of the candidate prefix *other<sub>v</sub>* nodes. We start with optimal  $k$ -node summary trees for  $T_{v_1}$ , for  $1 \leq k \leq K - 1$ . Inductively, suppose that we have computed (the entropy of) optimal  $k$ -node summary forests for  $T_{v_1}, \dots, T_{v_l}$ . We find optimal  $k$ -node summary forests for  $T_{v_1}, \dots, T_{v_l}, T_{v_{l+1}}$  as follows. For  $k = 1$ , the forest comprises a single *other<sub>v</sub>* node. For each  $k > 1$ , we choose the highest entropy among the following options: an optimal  $h$ -node summary forest for  $T_{v_1}, \dots, T_{v_l}$  plus an optimal  $(k - h)$ -node summary tree for  $T_{v_{l+1}}$ , for  $1 \leq h < k$ .

The correctness of this procedure is immediate: for  $k = 1$  clearly the only summary forest is a one-node forest. For  $k > 1$ ,  $T_{v_{l+1}}$  cannot be represented by the *other<sub>v</sub>* node (since we are discussing the handling of the *prefix other<sub>v</sub>* nodes) and so it must be represented by one tree in the summary forest; this implies that  $T_{v_1}, T_{v_2}, \dots, T_{v_l}$  must also be represented by one or more trees in the summary forest. Of course, the representation of each of the parts must be optimal. Our algorithm considers all possible ways of partitioning the nodes in the summary forest among these two parts; consequently it finds an optimal forest.

The process when  $v_j$  is the non-prefix node in *other<sub>v</sub>* is essentially identical. There are two changes: (i) *other<sub>v</sub>* is initialized to contain  $T_{v_j}$  (rather than being the empty set) and (ii) the incremental sweep skips tree  $T_{v_j}$ . The correctness argument is as in the previous paragraph.

Finally, to obtain optimal  $k$ -node summary forests for  $T_{v_1}, T_{v_2}, \dots, T_{v_{d_v}}$  one simply takes the best among the  $k$ -node forests computed for the different classes of candidate *other<sub>v</sub>* nodes. Again, correctness is immediate.

**Theorem 3.** *The running time of the algorithm is  $O(K^2n + n \log n)$ .*

NOTE. Our time bound is  $O(K^2n + n \log n)$  to build  $K$  maximum-entropy summary trees, or  $O(Kn + (n \log n)/K)$  amortized time for each. There is an obvious lower bound of  $\Omega(n + K^2)$  to build all  $K$  trees, since one has to read an  $n$ -node tree and produce trees having  $1, 2, 3, \dots, K$  nodes. Hence there cannot be a  $O(n)$ -time algorithm that generates all  $K$  trees, since it would violate the lower bound when  $K$  is  $\omega(\sqrt{n})$ . Of course, conceivably there is a linear-time algorithm to build a maximum-entropy  $k$ -node summary tree for a single value of  $k$ .

*Proof.* The running time is the sum of three terms:

- (1)  $O(n \log n)$ , for sorting the children of all nodes by size.
- (2)  $O(Kn)$  for initializations. In fact, the initializations for node  $v$  take time  $O(K \cdot \min\{d_v, K - 1\})$ , which is  $O(Kn)$  time in total.

(3) For each node  $v$ , the cost of processing node  $v_l$  when processing each of the classes of candidate  $other_v$  nodes. Let  $\langle v_a, v_{a+1}, \dots, v_{v_d} \rangle$  be the sequence of nodes processed when considering the candidate prefix  $other_v$  sets (nodes  $v_1, \dots, v_{a-1}$  are the nodes guaranteed to be in  $other_v$ ). When processing the near-prefix candidate  $other_v$  sets with non-prefix element  $v_j$ , the same sequence will be processed except that  $v_j$  will be omitted. For the class of prefix candidate sets, the cost for processing  $v_{l+1}$ , for  $a \leq l < v_d$ , is  $\min\{K-1, n_{v_a} + n_{v_{a+1}} + \dots + n_{v_l}\} \cdot \min\{K-1, n_{v_{l+1}}\} \leq \min\{K-1, n_{v_1} + n_{v_2} + \dots + n_{v_l}\} \cdot \min\{K-1, n_{v_{l+1}}\}$ , for we are seeking  $k$ -node summary forests for  $1 \leq k \leq K-1$ , and the number of nodes in a summary tree cannot be more than the number of nodes available in the relevant subtrees of  $T$ . The same bound applies for each of the remaining classes of candidate  $other_v$  sets and there are at most  $K-1$  of these classes. Since the number of child nodes being processed when computing at node  $v$  is  $d_v - a + 1 \leq d_v$ , the obvious upper bound here is  $O(K^3 \cdot d_v)$ . Summed over all  $v$ , this totals  $O(K^3 \cdot n)$ . However, Corollary 1 below shows that  $\sum_{\text{non-leaf } v} \sum_l \min\{n_{v_1} + n_{v_2} + \dots + n_{v_l}, K\} \cdot \min\{n_{v_{l+1}}, K\} \leq 2Kn$ , giving an overall time of  $O(n \log n + K^2 n)$ . ■

## 5 A Lemma For Running Time Analysis

In this section we state a lemma underlying the running time analysis of both the greedy algorithm and the exact algorithm. Let  $n$  be a positive integer and let  $T$  be a rooted,  $n$ -node tree, and *for this section only*, let  $v_1, v_2, \dots, v_{d_v}$  be  $v$ 's children in *any* order.

**Definition 3.** *Relative to  $T$ , let  $cost(v)$  be defined for all  $v \in T$  as follows. If  $v$  is a leaf,  $cost(v) = 0$ . If  $v$  is not a leaf,  $cost(v) = [\sum_{i=1}^{d_v} cost(v_i)] + [\sum_{i=1}^{d_v-1} \min\{n_{v_1} + n_{v_2} + \dots + n_{v_i}, K\} \cdot \min\{n_{v_{i+1}}, K\}]$ .*

**Lemma 4.** *For all  $v$ ,  $cost(v) \leq n_v^2$  if  $n_v \leq K$ , and  $cost(v) \leq 2Kn_v - K^2$ , if  $n_v > K$ .*

*Proof.* We prove the lemma by induction on the height of  $v$  (i.e., the maximum length of a path from  $v$  down to a leaf).

Basis. If  $v$  has height 0, i.e.,  $v$  is a leaf, then  $cost(v) = 0$ , whereas  $n_v = 1 \leq K$  and indeed  $cost(v) = 0 \leq 1 = n_v^2$ .

Inductive step. Let  $h \geq 0$  and assume that the statement is true for all nodes of height at most  $h$ .

To simplify the notation, we will use  $n_i$  to denote  $n_{v_i}$  and  $d$  to denote  $d_v$  from now on.

Let  $v$  be a node of height  $h+1$ ; then  $v$ 's children have height at most  $h$ . Therefore, by induction, if  $v$ 's children are  $v_1, v_2, \dots, v_d$ , then  $cost(v_i) \leq n_i^2$  if  $n_i \leq K$ , and  $cost(v_i) \leq 2Kn_i - K^2$ , if  $n_i > K$ .

Let  $Cost_j = \sum_{1 \leq i \leq j} cost(v_i) + [\sum_{i=1}^{j-1} \min\{n_1 + n_2 + \dots + n_i, K\} \cdot \min\{n_{i+1}, K\}]$ . We will show by induction on  $j$  that if  $\sum_{1 \leq i \leq j} n_i \leq K$ , then  $Cost_j \leq$

$(\sum_{1 \leq i \leq j} n_i)^2$ , and otherwise  $Cost_j \leq [2K \cdot \sum_{1 \leq i \leq j} n_i] - K^2$ , from which the result in the lemma is immediate.

Let  $t_j = \sum_{i=1}^j n_i$  and let  $u_j = n_{j+1}$ . There are five cases to consider.

i.  $t_j + u_j \leq K$ . Then

$$Cost_j \leq t_j^2 + u_j^2 + t_j u_j \leq (t_j + u_j)^2 = \left(\sum_{i=1}^{j+1} n_i\right)^2.$$

ii.  $t_j > K$  and  $u_j \leq K$ . Then

$$\begin{aligned} Cost_j &\leq (2Kt_j - K^2) + u_j^2 + Ku_j \leq 2Kt_j - K^2 + 2Ku_j \\ &\leq 2K(t_j + u_j) - K^2 = [2K \sum_{i=1}^{j+1} n_i] - K^2. \end{aligned}$$

iii.  $t_j \leq K$  and  $u_j > K$ .

This has essentially the same analysis as Case ii.

iv.  $t_j > K$  and  $u_j > K$ . Then

$$\begin{aligned} Cost_j &\leq (2Kt_j - K^2) + (2Ku_j - K^2) + K^2 \\ &\leq 2K(t_j + u_j) - K^2 = [2K \sum_{i=1}^{j+1} n_i] - K^2. \end{aligned}$$

v.  $t_j \leq K$ ,  $t_j + u_j > K$ , and  $u_j \leq K$ .

Let  $\Delta = t_j + u_j - K$ . Then

$$\begin{aligned} Cost_j &\leq t_j^2 + u_j^2 + t_j u_j = t_j^2 + (K + \Delta - t_j)^2 + t_j(K + \Delta - t_j) \\ &\leq t_j^2 - Kt_j - \Delta t_j + K^2 + \Delta^2 + 2K\Delta \\ &= 2(\Delta + K)K - K^2 + \Delta^2 - \Delta t_j + t_j^2 - Kt_j \\ &\leq 2(\Delta + K)K - K^2 \leq 2K \sum_{i=1}^{j+1} n_i - K^2. \end{aligned}$$

The next-to-last inequality follows because  $\Delta \leq t_j$  and  $t_j \leq K$ . ■

**Corollary 1.** For  $K \geq 1$ ,  $\sum_{non-leaf\ v} [\sum_{i=1}^{d_v-1} \min\{n_{v_1} + n_{v_2} + \dots + n_{v_i}, K\} \cdot \min\{n_{v_{i+1}}, K\}] \leq 2Kn$ .

*Proof.* Summing over all non-leaf nodes in Definition 3 yields that the term we are bounding equals  $cost(r)$ , where  $r$  is the root of the tree; the result now follows from Lemma 4. ■

## 6 Greedy Algorithm

The greedy algorithm proposed in [4] is precisely the algorithm proposed herein for the exact solution but with the *other* sets restricted to being prefix sets. In [4] Greedy was shown to run in time  $O(K^2n + n \log n)$ . Here, we shave off a factor of  $K$  from the first term.

**Corollary 2.** *(of Lemma 4). The time needed by the greedy algorithm to generate summary trees of orders  $k = 1, 2, \dots, K$  is  $O(Kn + n \log n)$ .*

*Proof.* Aside from initializations (which take time  $O(Kn)$ ) and sorting (which takes time  $O(n \log n)$ ), the time needed by the greedy algorithm is  $O$  of

$$\sum_v \left[ \sum_{i=1}^{d_v-1} \min\{n_{v_1} + n_{v_2} + \dots + n_{v_i}, K\} \cdot \min\{n_{v_{i+1}}, K\} \right],$$

which by Corollary 1 is at most  $2Kn$ , giving an overall bound of  $O(Kn + n \log n)$ . ■

Again, one can reduce the  $O(n \log n)$  term to  $O(n \log K)$ , giving an overall run time of  $O(nK)$ . Here we rely on the fact that the greedy algorithm will necessarily put  $v_1, v_2, \dots, v_{d_v-K}$  into *other<sub>v</sub>*. To save time, we can modify Greedy so as to put those nodes into *other<sub>v</sub>* and only individually process children  $v_{d_v-K+1}, v_{d_v-K+2}, \dots, v_{d_v}$ . We can find the  $d_v - K$  children of  $v$  of least size via a selection (not sorting) algorithm and then sort only the remaining  $K$  children. This makes the total sorting time over all nodes  $v$   $O$  of  $\sum_v \min\{d_v, K\} \log(\min\{d_v, K\}) \leq \sum_v \min\{d_v, K\} \log K \leq (\log K) \sum_v d_v = n \log K$ .

## 7 Improved Approximation Algorithm

In this section we describe an algorithm that computes an approximately entropy-optimal  $k$ -node summary tree. Our algorithm relies on the following outline from [4]:

1. One can rescale the weights in a tree to make them sum up to any positive integral value  $W_0$ , while leaving the entropy of any summary tree unchanged. (This is obvious.)
2. One can use algorithmic discrepancy theory to round each resulting real node weight  $w_v$  to value  $w'_v$  equal to either  $\lfloor w_v \rfloor$  or  $1 + \lfloor w_v \rfloor$  such that for each node  $v \in T$ ,  $|\sum_{u \in T_v} w'_u - \sum_{u \in T_v} w_u| \leq 1$  for all  $v$  simultaneously, without changing the overall sum.
3. Using Naudts's theorem [8] that almost identical probability distributions have almost identical entropy, one can prove, for some integer  $W_0$  which is  $O((K/\epsilon) \log(K/\epsilon))$ , that if one finds a maximum entropy summary tree  $T^*$  for the modified weights  $(w'_v)$ , then  $T^*$  has entropy (measured according to the *original* weights  $w_v$ ) at most  $\epsilon$  less than that of the truly maximum entropy summary tree.

Suppose that the weights on  $T$  are integral and sum to  $W_0$ . Clearly the number of nodes of positive weight cannot exceed  $W_0$ ; however, the 0-weight nodes could far outnumber the positive-weight nodes. Indeed, that is exactly what happens if  $n \gg W_0$ .

Our algorithm exploits the fact that little processing is needed for most of the 0-weight nodes. In fact, we will need to compute summary trees for only the non-zero weight nodes and for at most  $2(W_0 - 1)$  0-weight nodes.

The algorithm works with a tree  $T'$ , a reduced version of  $T$  in which some 0-weight nodes have been removed. The following notation will be helpful.  $F_T(v, k)$  denotes the maximum pseudo-entropy of a  $k$ -node summary tree of  $T_v$ , where  $T_v$  is a subtree of tree  $T$ ; similarly,  $F_{T'}(v, k)$  denotes the maximum pseudo-entropy of a  $k$ -node summary tree of  $T'_v$ , where  $T'_v$  is a subtree of tree  $T'$ .

$T'$  is obtained from  $T$  as follows: for each positively-sized node  $v$  in  $T$ , if  $v$  has one or more size-0 children, remove them and their descendants and replace them all by a single 0-weight child. Clearly optimal summary trees in  $T'$  form optimal summary trees in  $T$  (for the only difference in summarizing  $T$  is that we could add 0-weight nodes no longer present in  $T'$ , and these would contribute 0 to the entropy). Note that if  $v$  is a 0-weight non-leaf node in  $T'$  then it must have non-zero size (assuming  $T$  has at least one positive-weight node). The following result is immediate.

**Lemma 5.** *Let  $T$  have  $n$  nodes and  $T'$  have  $n'$  nodes. Let  $v$  be a node in  $T'$  with  $n(v)$  descendants in  $T$  and  $n'(v)$  descendants in  $T'$ . Then  $F_T(v, k) = F_{T'}(v, k)$  for  $1 \leq k \leq n'(v)$ . For  $n'(v) + 1 \leq k \leq n$ ,  $F_T(v, k) = F_{T'}(v, n'(v))$ .*

Note that  $F_{T'}(v, n'(v))$  is attained by a partition of the set of  $v$ 's children in  $T'$  into singletons.

(Now of course we *have* changed the problem, since  $T'$  might have fewer than  $K$  nodes. However, if this happens, then optimal summary trees of  $T$  having more than  $|T'|$  nodes have no more entropy than optimal summary trees of  $T$  having exactly  $|T'|$  nodes.)

Even after the reduction it may be the case that  $|T'| \gg W_0$ , for  $T'$  might still contain long paths of 0-weight nodes in which each node has only one positively-sized child. However, the following lemmas show that they add little to the cost of computing optimal summary trees.

**Lemma 6.** *Let  $v$  be a 0-weight node in  $T'$  with a single child  $u$ . Then for  $2 \leq k \leq |T'_v|$ ,  $F_{T'}(v, k + 1) = F_{T'}(u, k)$ ; also  $F_{T'}(v, 1) = F_{T'}(u, 1)$ .*

*Proof.* For  $k \geq 2$ , the  $(k + 1)$ -node summary tree for  $T'_v$  adds a zero-weight node  $\{v\}$  to the  $k$ -node summary tree for  $T'_u$ . For  $k = 1$  both trees have a single node of weight  $w_u$ .

**Lemma 7.** *Let  $v$  be a 0-weight node in  $T'$  with exactly two children, a 0-weight leaf  $v_1$  and a child  $u$  of positive size. Then for  $3 \leq k \leq |T'_v|$ ,  $F_{T'}(v, k + 2) = F_{T'}(u, k)$ ; also  $F_{T'}(v, 2) = F_{T'}(v, 1) = F_{T'}(u, 1)$ .*

The proof of this lemma is essentially the same as that of Lemma 6. The following corollary is immediate.

**Corollary 3.** *Let  $v_1, v_2, \dots, v_l$ , for  $l > 1$ , be a descending path of 0-weight nodes in  $T'$  such that each  $v_i$ ,  $1 \leq i \leq l$  either has one child, or has exactly two children one of which is a 0-weight leaf. Further suppose that  $l'$  of these nodes are in the second category. Node  $v_l$  must have a child of positive size (as otherwise  $v_l \neq v_1$  would be a size-0 non-leaf). Let  $u$  be the child of  $v_l$  of positive size. Then for  $1 \leq k \leq |T'_{v_1}| - (l + l')$ ,  $F_{T'}(v_1, k + l + l') = F_{T'}(u, k)$ ; and for  $j \leq l + l'$ ,  $F_{T'}(v_1, j) = F_{T'}(u, 1)$ .*

This corollary implies that given the entropies of optimal entropy summary trees at a node  $u$  at the bottom of a maximal path of 0-weight nodes, one can obtain the entropies of the optimal entropy summary trees at node  $v_1$  at the top of the path in time  $O(K)$ .

At the remaining nodes in  $T'$  we perform the same computation as in the exact algorithm. As we can show, there are  $O(W_0)$  such nodes, which leads to the following running time bound.

**Theorem 4.** *The approximation algorithm to obtain a summary tree that has entropy within an additive  $\epsilon$  of that of the optimal summary tree runs in time  $O(n + W_0 \cdot K^3)$ , where  $W_0 = O((K/\epsilon) \log(K/\epsilon))$ .*

*Proof.* We begin by bounding the numbers of nodes of various types. Clearly, there are at most  $W_0$  non-zero weight nodes. Thus there are at most  $(W_0 - 1)$  0-weight nodes with two or more non-zero weight subtrees. All other 0-weight nodes are either leaf nodes or lie on maximal paths of 0-weight nodes with one non-zero weight subtree. Further, there can be at most  $W_0 - 1$  such maximal paths.

The naive bound on the cost of the computation at a node  $v$  in the exact algorithm is  $O(K^3 \cdot d_v)$  (see the proof of Theorem 3), and this applies to the non-zero weight nodes and the 0-weight nodes with two or more non-zero weight subtrees, giving a cost of  $O(K^3 \cdot W_0)$  in total.

The cost of processing the maximal paths of 0 weight is  $O(W_0 \cdot K)$ .

Finally, recall that we need to sort the children in non-decreasing order by size for each parent node with non-zero weight or with two or more non-zero weight subtrees. We implement this by means of a radix sort on the pairs (parent,  $W_i$ ), over these parent nodes (recall that  $W_i$  is the size of the  $i$ th child). There are  $O(W_0)$  such pairs, with indices in the ranges  $n$  and  $W_0$  respectively, yielding a running time of  $O(n + W_0)$  for the radix sort. ■

In contrast to the  $O(nK^2 + n \log n)$  bound for the exact algorithm, here the sophisticated analysis of Lemma 4 cannot be applied. The reason is that Lemma 4 assumes that a tree of  $r$  nodes yields at most  $r$  optimal summary trees each having a distinct entropy (so if the  $k$ -node and  $(k + 1)$ -node optimal trees have the same entropy only one of them is counted). However, the same claim fails to hold for trees having  $r$  *positive-weight* nodes (and a total number of nodes potentially vastly exceeding  $r$ ), as would be needed in order to apply Lemma 4 here.



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